Algorithms: Dynamic Programming (Matrix Chain Multi. and Longest Common Subsequence)

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DYNAMIC PROGRAMMING

(An algorithmic paradigm not a way of "programming")

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What is $2^5 + 3 - \sqrt{16}$?

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Parenthesization	Cost computation	Cost
$A \times ((B \times C) \times D)$	$20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100$	$120,200$
$(A \times (B \times C)) \times D$	$20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$	$60,200$
$(A \times B) \times (C \times D)$	$50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$	$7,000$

MATRIX-CHAIN MULTIPLICATION

Cost of Matrix Multiplication

$$A_{p,q} \times B_{q,r}$$

Cost of Matrix Multiplication

$$\underset{p}{\overset{q}{A_{p,q}}} \times \underset{q}{\overset{r}{B_{q,r}}}$$

$$\begin{matrix} p \\ \left\{ \right. \end{matrix} \overbrace{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,q) \\ (2,1) & (2,2) & \cdots & (2,q) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,q) \end{bmatrix}}^{q} \qquad \begin{matrix} q \\ \left\{ \right. \end{matrix} \overbrace{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (q,1) & (q,2) & \cdots & (q,r) \end{bmatrix}}^{r}$$

Cost of Matrix Multiplication

$$\underbrace{A_{p,q}}_{q} \times \underbrace{B_{q,r}}_{r}$$

$$p\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,q) \\ (2,1) & (2,2) & \cdots & (2,q) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,q) \end{bmatrix}\right. \qquad q\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (q,1) & (q,2) & \cdots & (q,r) \end{bmatrix}\right.$$

$$\Downarrow$$

$$\underbrace{C_{p,r}}_{r}$$

$$p\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,r) \end{bmatrix}\right.$$

Cost of Matrix Multiplication

$$
\underbrace{A_{p,q}}_{q}
\begin{matrix} \\ p \end{matrix}
\begin{bmatrix}
(1,1) & (1,2) & \cdots & (1,q) \\
(2,1) & (2,2) & \cdots & (2,q) \\
\vdots & \vdots & \ddots & \vdots \\
(p,1) & (p,2) & \cdots & (p,q)
\end{bmatrix}
\times
\underbrace{B_{q,r}}_{r}
\begin{matrix} \\ q \end{matrix}
\begin{bmatrix}
(1,1) & (1,2) & \cdots & (1,r) \\
(2,1) & (2,2) & \cdots & (2,r) \\
\vdots & \vdots & \ddots & \vdots \\
(q,1) & (q,2) & \cdots & (q,r)
\end{bmatrix}
$$

$$
\Downarrow
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\underbrace{C_{p,r}}_{r}
\begin{matrix} \\ p \end{matrix}
\begin{bmatrix}
(1,1) & (1,2) & \cdots & (1,r) \\
\boxed{(2,1)} & (2,2) & \cdots & (2,r) \\
\vdots & \vdots & \ddots & \vdots \\
(p,1) & (p,2) & \cdots & (p,r)
\end{bmatrix}
$$

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p\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,r) \end{bmatrix}\right.
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\underbrace{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,q) \\ (2,1) & (2,2) & \cdots & (2,q) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,q) \end{bmatrix}}_{q} \Big\}\, p \quad A_{p,q}
$$

$$
\times
$$

$$
\underbrace{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (q,1) & (q,2) & \cdots & (q,r) \end{bmatrix}}_{r} \Big\}\, q \quad B_{q,r}
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\underbrace{A_{p,q}}_{q}\ p\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,q) \\ (2,1) & (2,2) & \cdots & (2,q) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,q) \end{bmatrix}\right. \times \underbrace{B_{q,r}}_{r}\ q\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (q,1) & (q,2) & \cdots & (q,r) \end{bmatrix}\right.
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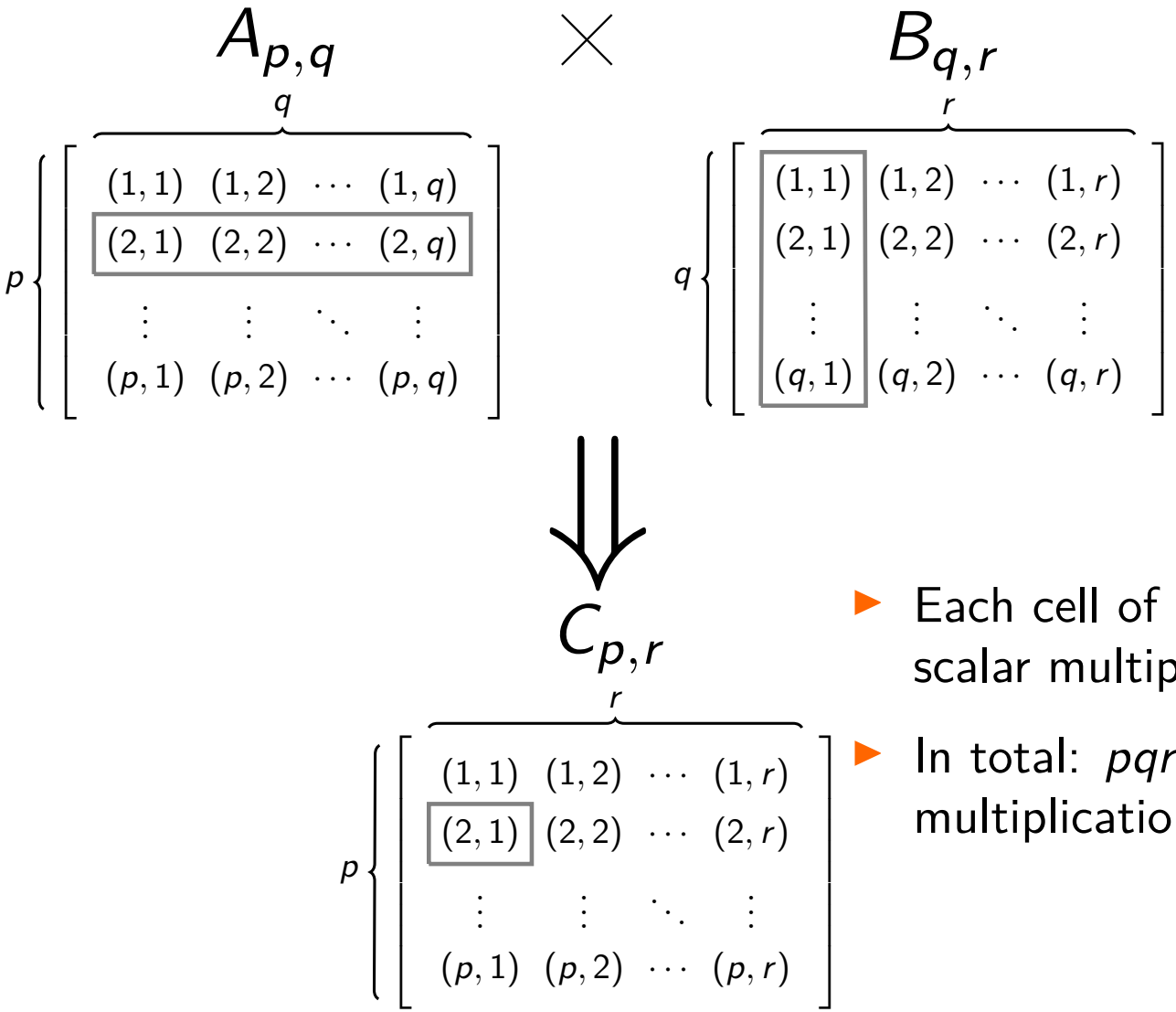
\underbrace{C_{p,r}}_{r}\ p\left\{\begin{bmatrix} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,r) \end{bmatrix}\right.
$$

- Each cell of C requires q scalar multiplications.

Cost of Matrix Multiplication



- Each cell of C requires q scalar multiplications.
- In total: pqr scalar multiplications.

Cost of Matrix Multiplication

$$A_{p,q} \times B_{q,r} \Rightarrow C_{p,r}$$

$$A_{p,q} = \begin{array}{c} \overbrace{}^{q} \\ \left\{ p \right. \left[\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,q) \\ (2,1) & (2,2) & \cdots & (2,q) \\ \vdots & \vdots & \ddots & \vdots \\ (p,1) & (p,2) & \cdots & (p,q) \end{array} \right] \end{array}$$

$$B_{q,r} = \begin{array}{c} \overbrace{}^{r} \\ \left\{ q \right. \left[\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,r) \\ (2,1) & (2,2) & \cdots & (2,r) \\ \vdots & \vdots & \ddots & \vdots \\ (q,1) & (q,2) & \cdots & (q,r) \end{array} \right] \end{array}$$

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- Each cell of C requires q scalar multiplications.
- In total: pqr scalar multiplications.
- The scalar multiplications dominate the time complexity.

Matrix Chain Multiplication

Definition

Input: A chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices, where for $i = 1, 2, \ldots, n$, matrix A_i has dimension $p_{i-1} \times p_i$.

Output: A full parenthesization of the product $A_1 A_2 \cdots A_n$ in a way that minimizes the number of scalar multiplications.

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Remarks

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Remarks

- We are not asked to calculate the product, only find the best parenthesization.

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Remarks

- We are not asked to calculate the product, only find the best parenthesization.
- The parenthesization can significantly affect the number of multiplications.

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Example

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Example

- A product $A_1 A_2 A_3$ with dimensions: 50×5, 5×100 and 100×10.

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Example

- A product $A_1 A_2 A_3$ with dimensions: 50×5, 5×100 and 100×10.
- Calculating $(A_1 A_2) A_3$ requires: $50 \cdot 5 \cdot 100 + 50 \cdot 100 \cdot 10 = 75000$ scalar multiplications.

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Example

- A product $A_1 A_2 A_3$ with dimensions: 50×5, 5×100 and 100×10.
- Calculating $(A_1 A_2) A_3$ requires: $50 \cdot 5 \cdot 100 + 50 \cdot 100 \cdot 10 = 75000$ scalar multiplications.
- Calculating $A_1 (A_2 A_3)$ requires: $5 \cdot 100 \cdot 10 + 50 \cdot 5 \cdot 10 = 7500$ scalar multiplications.

Optimal Substructure

Theorem

If:

- *the outermost parenthesization in an optimal solution is:* $(A_1 A_2 \cdots A_i)(A_{i+1} A_{i+2} \cdots A_n)$.
- P_L *and* P_R *are optimal parenthesizations for* $A_1 A_2 \cdots A_i$ *and* $A_{i+1} A_{i+2} \cdots A_n$, *respectively.*

Then, $((P_L) \cdot (P_R))$ *is an optimal parenthesizations for* $A_1 A_2 \cdots A_n$.

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Then, $((P_L) \cdot (P_R))$ *is an optimal parenthesizations for* $A_1 A_2 \cdots A_n$.

Proof

- Let $((O_L) \cdot (O_R))$ be an optimal parenthesization, where O_L and O_R are parenthesizations for $A_1 A_2 \cdots A_i$ and $A_{i+1} \cdots A_n$, respectively.

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Proof

- Let $((O_L) \cdot (O_R))$ be an optimal parenthesization, where O_L and O_R are parenthesizations for $A_1 A_2 \cdots A_i$ and $A_{i+1} \cdots A_n$, respectively.
- Let $M(P)$ be the number of scalar multiplications required by a parenthesization.

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Then, $((P_L) \cdot (P_R))$ *is an optimal parenthesizations for* $A_1 A_2 \cdots A_n$.

Proof

$$M((O_L) \cdot (O_R))$$

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Then, $((P_L) \cdot (P_R))$ *is an optimal parenthesizations for* $A_1 A_2 \cdots A_n$.

Proof

$$M((O_L) \cdot (O_R)) = p_0 \cdot p_i \cdot p_n + M(O_L) + M(O_R)$$

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Proof

$$M((O_L) \cdot (O_R)) = p_0 \cdot p_i \cdot p_n + M(O_L) + M(O_R)$$

- Since P_L and P_R are optimal: $M(P_L) \leq M(O_L)$ and $M(P_R) \leq M(O_R)$.

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Proof

$$\begin{aligned} M((O_L) \cdot (O_R)) &= p_0 \cdot p_i \cdot p_n + M(O_L) + M(O_R) \\ &\geq p_0 \cdot p_i \cdot p_n + M(P_L) + M(P_R) \end{aligned}$$

- Since P_L and P_R are optimal: $M(P_L) \leq M(O_L)$ and $M(P_R) \leq M(O_R)$.

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- Since P_L and P_R are optimal: $M(P_L) \leq M(O_L)$ and $M(P_R) \leq M(O_R)$.

Recursive Formula

- Let $m[i, j]$ be the optimal number of scalar multiplications for calculating $A_i A_{i+1} \cdots, A_j$.

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- Let $m[i,j]$ be the optimal number of scalar multiplications for calculating $A_i A_{i+1} \cdots, A_j$.

- $m[i,j]$ can be expressed recursively as follows:

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \ , \\ \min_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j \ . \end{cases}$$

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- Each $m[i,j]$ depend only on subproblems with smaller $j - i$.
- A bottom-up algorithm should solve subproblems in increasing $j - i$ order.

Example

Instance

matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimensions	30×35	35×15	15×5	5×10	10×20	20×25

Bottom-Up Algorithm

```
MATRIX-CHAIN-ORDER(p)
1   n = p.length − 1
2   let m[1..n, 1..n] and s[1..n, 1..n] be new tables
3   for i = 1 to n
4       m[i, i] = 0
5   for ℓ = 2 to n                  // ℓ is the chain length
6       for i = 1 to n − ℓ + 1
7           j = i + ℓ − 1
8           m[i, j] = ∞
9           for k = i to j − 1
10              q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
11              if q < m[i, j]
12                  m[i, j] = q
13                  s[i, j] = k
14  return m and s
```

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⇐ s stores the optimal choice

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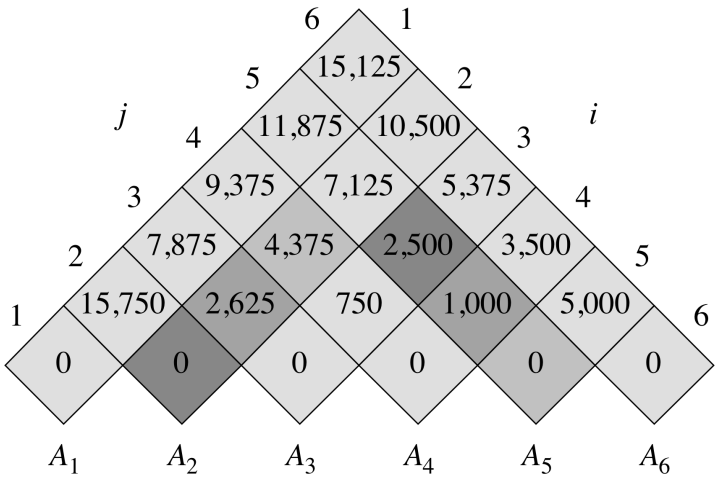
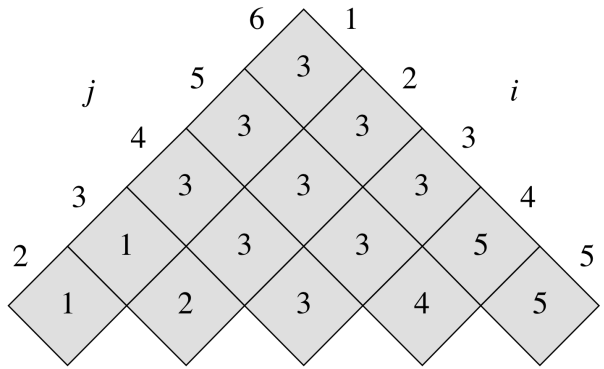
m



s



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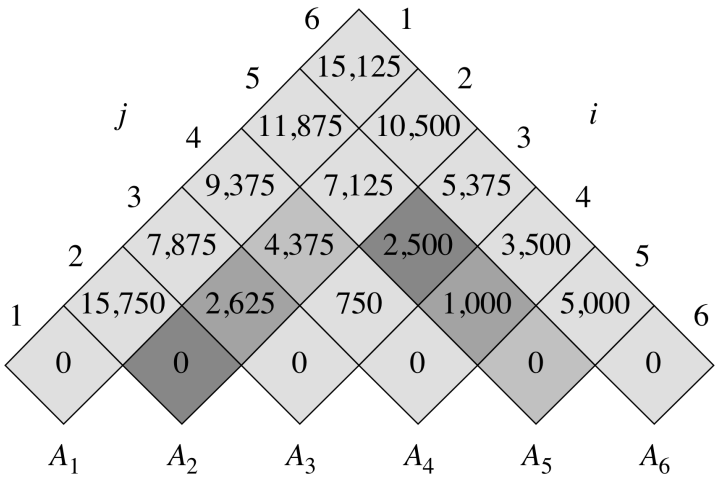
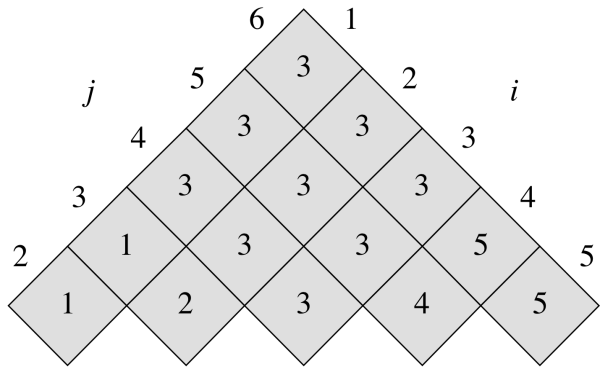
m



s



$A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6$

Example

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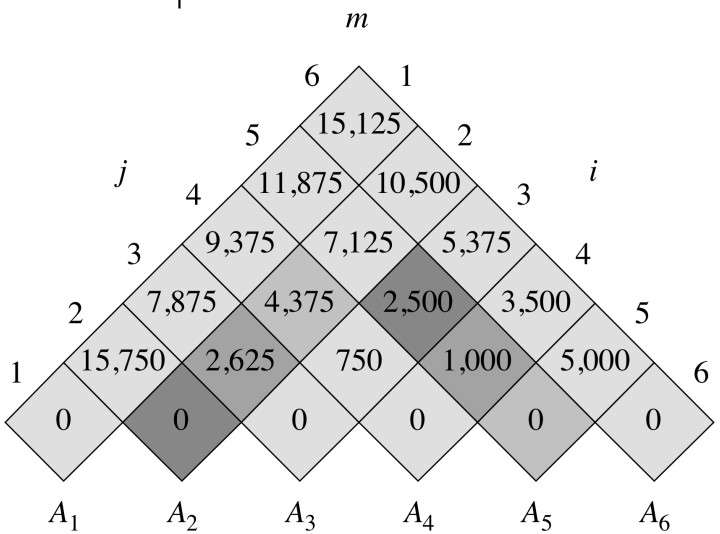

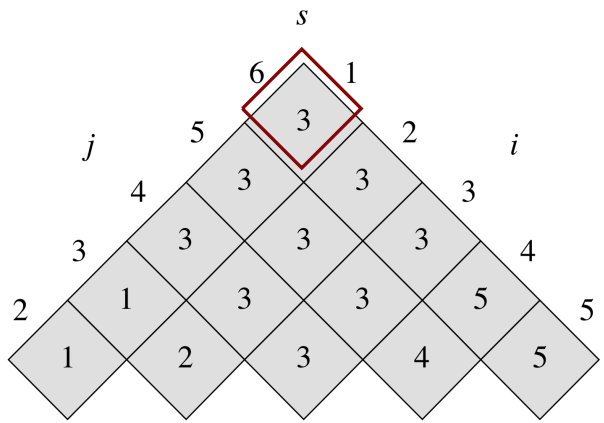

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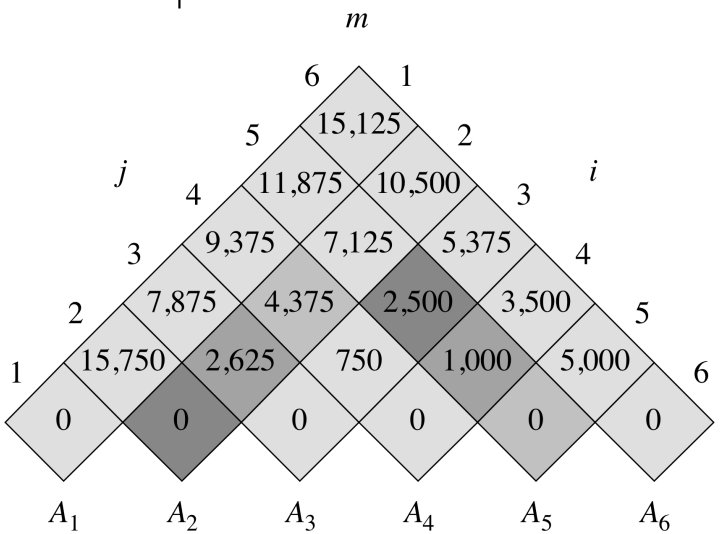

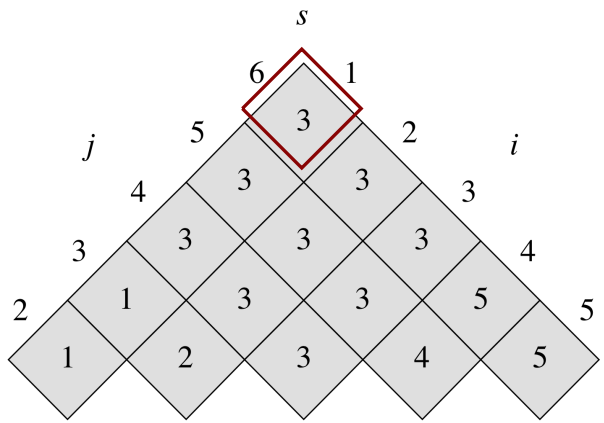

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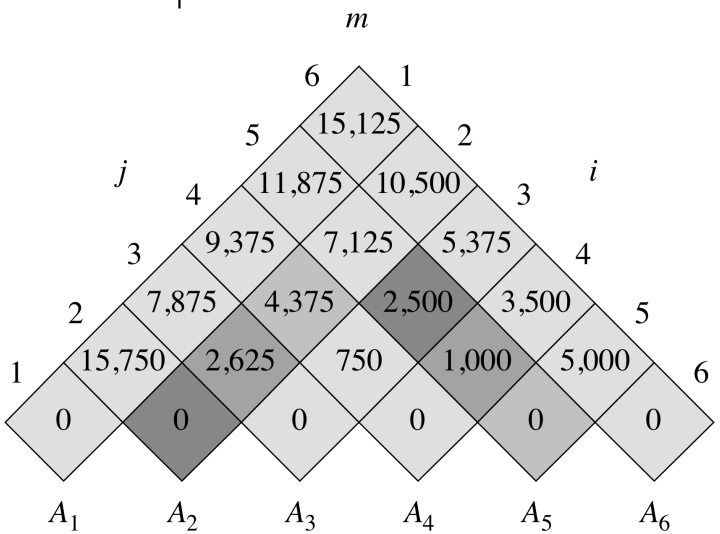

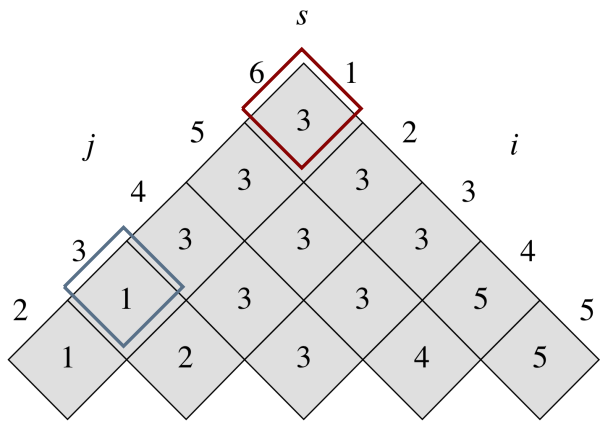

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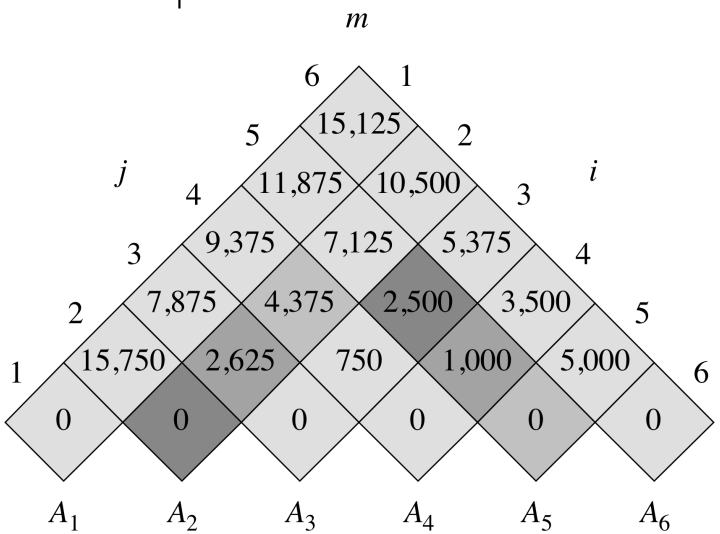

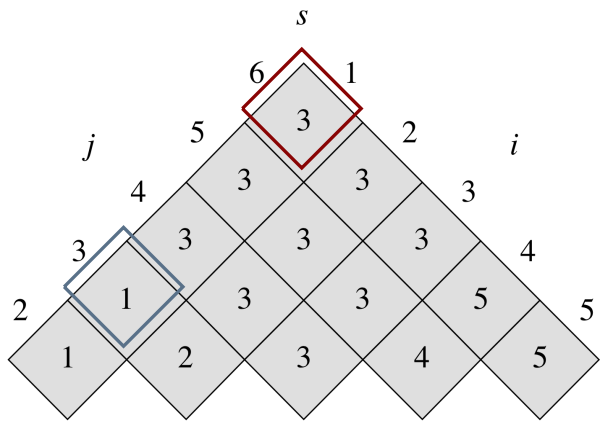

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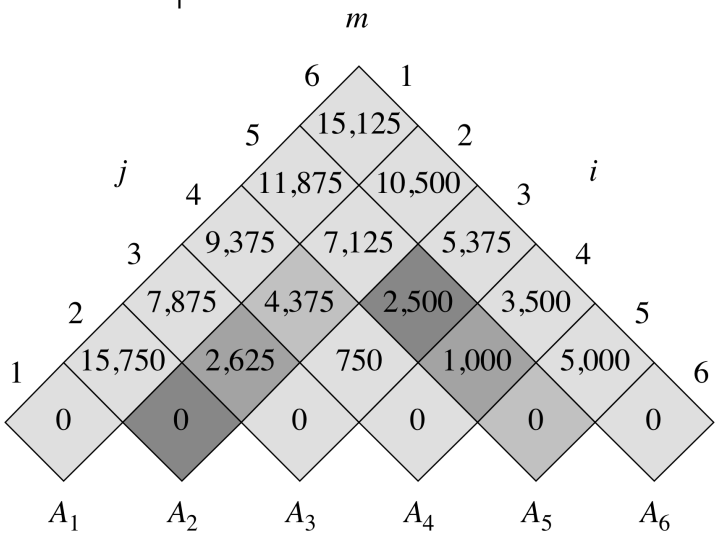

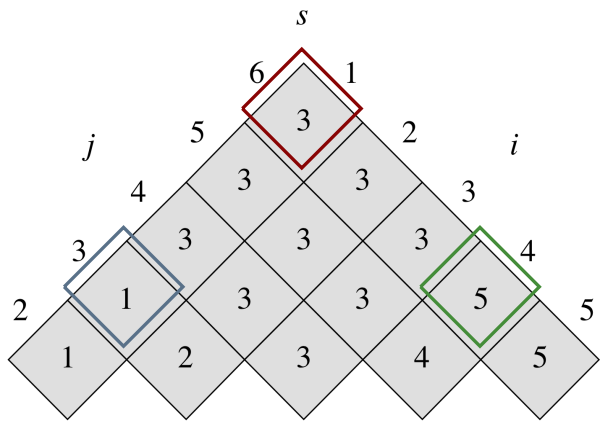

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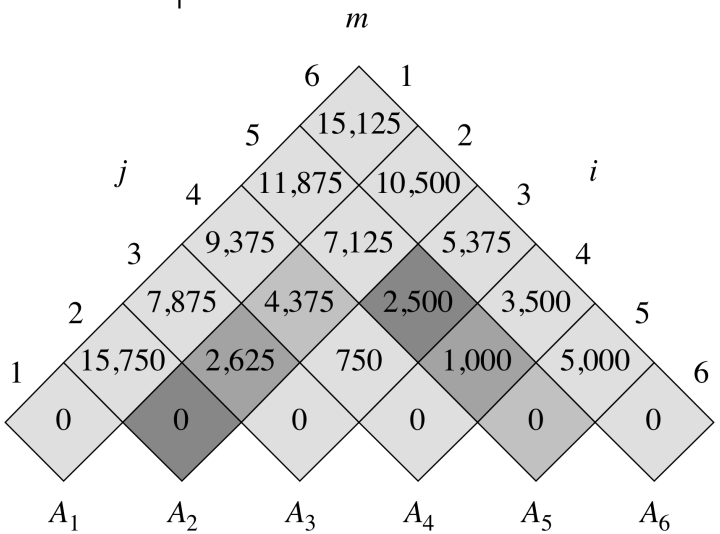

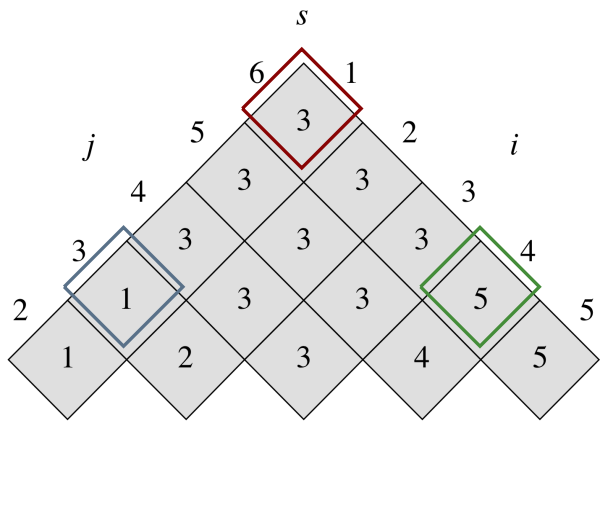

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Algorithm for Recovering an Optimal Solution

```
PRINT-OPTIMAL-PARENS(s, i, j)
1   if i == j
2       print "A_i"
3   else print "("
4       PRINT-OPTIMAL-PARENS(s, i, s[i, j])
5       PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
6       print ")"
```

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$$(A_1 \cdots A_k)(A_{k+1} \cdots A_n)$$

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Hence, if we let $m[i, j]$ be the optimal value for chain multiplication of matrices $A_i, \ldots, A_j$, we can express $m[i, j]$ recursively as follows

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\} & \text{otherwise if } i < j \end{cases}$$

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Overlapping subproblem: Solve recurrence using top-down with memoization or bottom-up which yields an algorithm that runs in time $\Theta(n^3)$.

LONGEST COMMON SUBSEQUENCE

Longest common subsequence

Definition

INPUT: 2 sequences, $X = \langle x_1, \ldots, x_m \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$.

OUTPUT: A subsequence common to both whose length is longest.

A subsequence doesn't have to be consecutive, but it has to be in order

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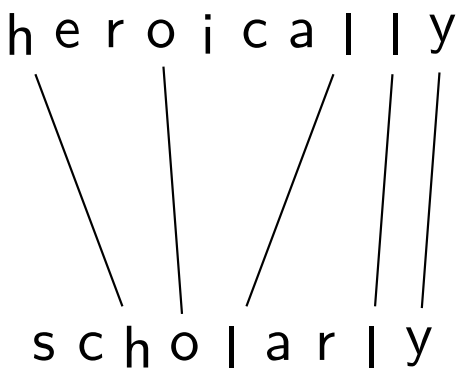

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No natural greedy algorithm for the problem :(

Dynamic programming comes to the rescue

Start at the end of both words and move to the left step-by-step

B A B D B A

D A C B C B A

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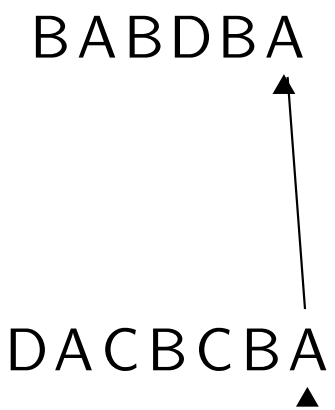

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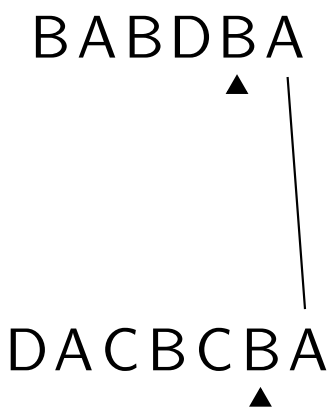

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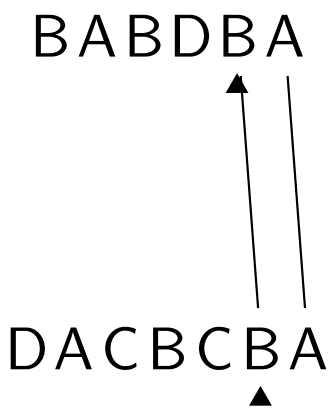

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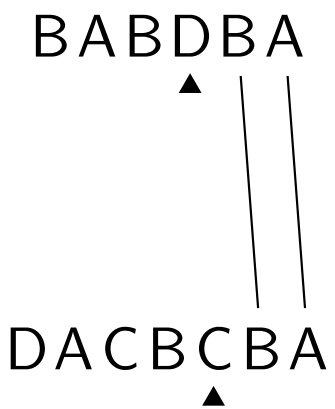

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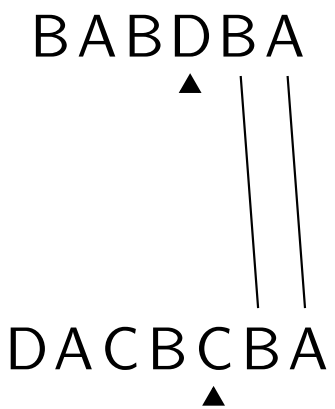

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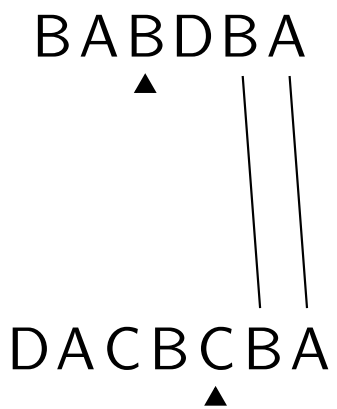

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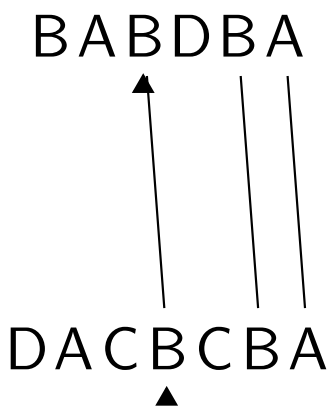

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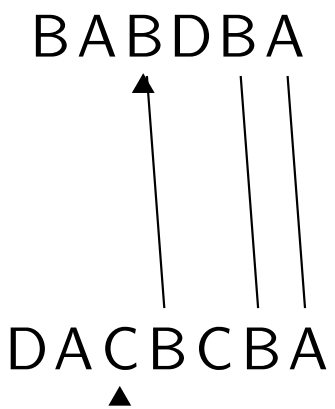

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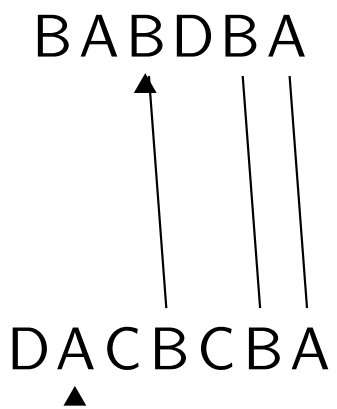

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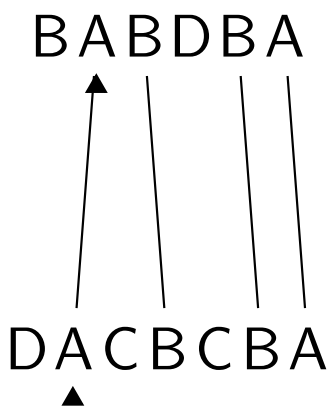

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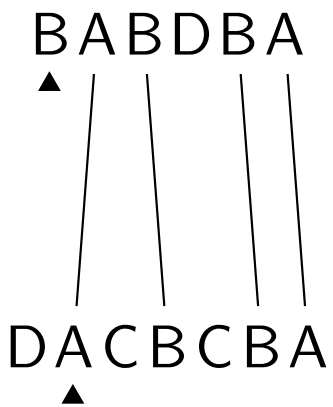

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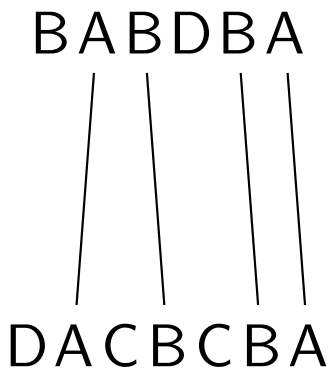

Optimal substructure

Let X_i and Y_j denote the prefixes $\langle x_1, x_2, \ldots, x_i \rangle$ and $\langle y_1, y_2, \ldots y_j \rangle$

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Theorem

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1. *If $x_i = y_j$ then $z_k = x_i = y_j$ and Z_{k-1} is an LCS of X_{i-1} and Y_{j-1}*

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Proof. Suppose $z_k \neq x_i = y_j$

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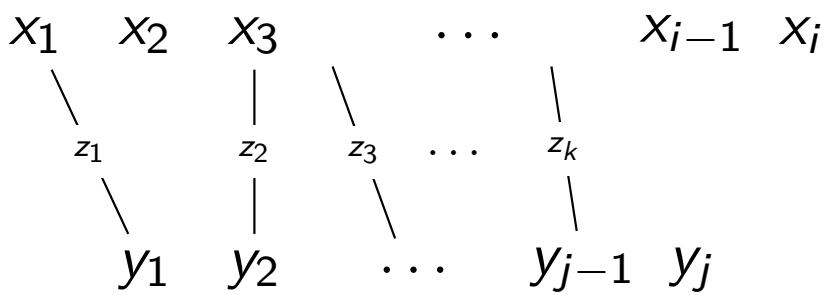

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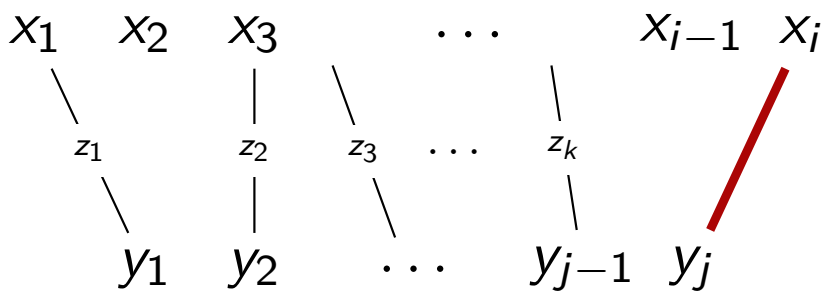
Proof. Suppose $z_k \neq x_i = y_j$ but then $Z' = \langle z_1, \ldots, z_k, x_i \rangle$ is a common subsequence of X_i and Y_j which contradicts Z being a LCS.



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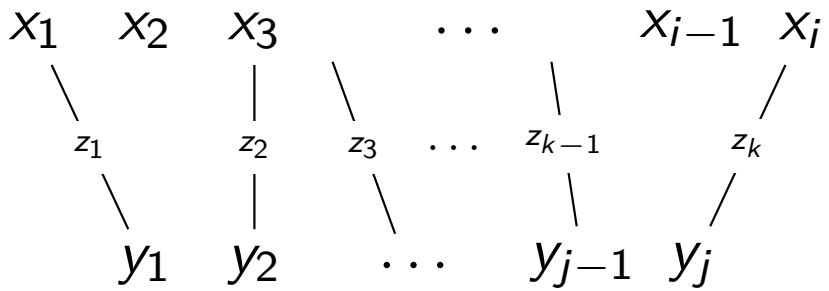

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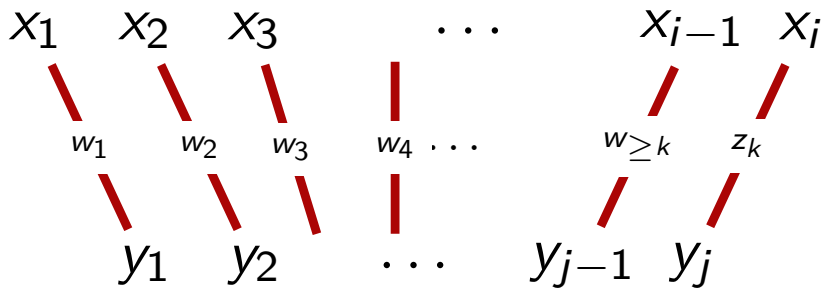

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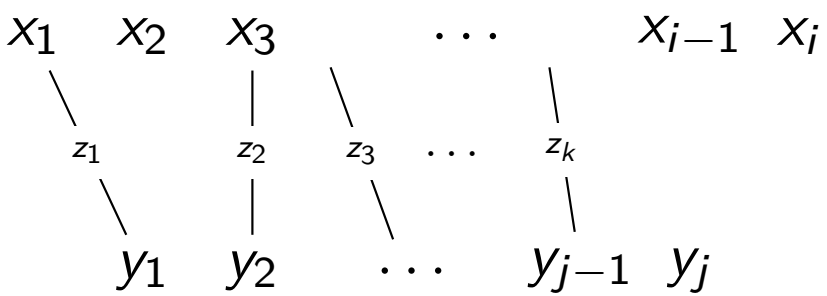
2. *If $x_i \neq y_j$, then $z_k \neq x_i \Rightarrow Z$ is an LCS of X_{i-1} and Y_j*

Proof. Z is a common subsequence to X_{i-1} and Y_j. Suppose Z is not a LCS to X_{i-1} and Y_j



Optimal substructure

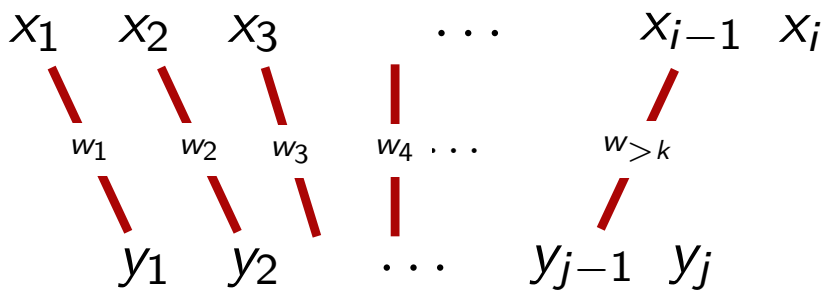
Let X_i and Y_j denote the prefixes $\langle x_1, x_2, \ldots, x_i \rangle$ and $\langle y_1, y_2, \ldots y_j \rangle$

Theorem

Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X_i and Y_j.

1. *If $x_i = y_j$ then $z_k = x_i = y_j$ and Z_{k-1} is an LCS of X_{i-1} and Y_{j-1}*
2. *If $x_i \neq y_j$, then $z_k \neq x_i \Rightarrow Z$ is an LCS of X_{i-1} and Y_j*

Proof. Z is a common subsequence to X_{i-1} and Y_j. Suppose Z is not a LCS to X_{i-1} and Y_j but then exists a common subsequence W of X_{i-1} and Y_j that has length $> k$ and, as it is also a common subsequence to X_i and Y_j, it contradicts the optimality of Z



Optimal substructure

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Theorem

Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X_i and Y_j.

1. *If $x_i = y_j$ then $z_k = x_i = y_j$ and Z_{k-1} is an LCS of X_{i-1} and Y_{j-1}*
2. *If $x_i \neq y_j$, then $z_k \neq x_i \Rightarrow Z$ is an LCS of X_{i-1} and Y_j*
3. *If $x_i \neq y_j$, then $z_k \neq y_j \Rightarrow Z$ is an LCS of X_i and Y_{j-1}*

Proof. Same argument as for (2).

From the above theorem, we know that the length of a LCS of X_i, Y_j is

$$1 + \text{LCS of } X_{i-1} \text{ and } Y_{j-1} \qquad \text{if } x_i = y_j$$

$$\text{either LCS of } X_{i-1}, Y_j \text{ or LCS of } X_i, Y_{j-1} \qquad \text{otherwise}$$

Recursive formulation

Define $c[i, j] =$ length of LCS of X_i and Y_j. We want $c[m, n]$

$$c[i, j] = \begin{cases}$$

Recursive formulation

Define $c[i, j] =$ length of LCS of X_i and Y_j. We want $c[m, n]$

$$c[i, j] = \begin{cases} & \text{if } i = 0 \text{ or } j = 0 \end{cases}$$

Recursive formulation

Define $c[i, j] =$ length of LCS of X_i and Y_j. We want $c[m, n]$

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \end{cases}$$

Recursive formulation

Define $c[i, j] =$ length of LCS of X_i and Y_j. We want $c[m, n]$

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ & \text{if } i, j > 0 \text{ and } x_i = y_j \end{cases}$$

Recursive formulation

Define $c[i, j] =$ length of LCS of X_i and Y_j. We want $c[m, n]$

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \end{cases}$$

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- Naive implementation solves same problems many many times

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								

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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0							
2	0							
3	0							
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5	0							
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2	0							
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1	0	0						
2	0							
3	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0					
2	0							
3	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0				
2	0							
3	0							
4	0							
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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1			
2	0							
3	0							
4	0							
5	0							
6	0							

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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1		
2	0							
3	0							
4	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	
2	0							
3	0							
4	0							
5	0							
6	0							

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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0							
3	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0						
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1					
3	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1				
3	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1			
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1		
3	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	
3	0							
4	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0						
4	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1					
4	0							
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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1				
4	0							
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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2			
4	0							
5	0							
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i \ j	0	1	2	3	4	5	6	7
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2		
4	0							
5	0							
6	0							

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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	
4	0							
5	0							
6	0							

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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0							
5	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1						
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1					
5	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1				
5	0							
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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2			
5	0							
6	0							

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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2		
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	
5	0							
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1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0							
6	0							

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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0							
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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1						
6	0							

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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1					
6	0							

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i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1				
6	0							

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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2			
6	0							

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0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2		
6	0							

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	
6	0							

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0							

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0							

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1						

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2					

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2	2				

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2	2	2			

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2	2	2	2		

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2	2	2	2	3	

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2	2	2	2	3	4

Bottom-up approach for LCS

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1
2	0	0	1	1	1	1	1	2
3	0	0	1	1	2	2	2	2
4	0	1	1	1	2	2	2	2
5	0	1	1	1	2	2	3	3
6	0	1	2	2	2	2	3	4

Longest common subsequence has length 4

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0							
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0							
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑						
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑					
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑				
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖			
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←		
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0							
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑						
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖					
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←				
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑			
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑		
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0							
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑						
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑					
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑				
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖			
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←		
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0							
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖						
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑					
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑				
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑			
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑		
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	
5	0							
6	0							

Recording optimal solution

Store optimal choices in an additional array $b[i, j]$

$X = \langle B, A, B, D, B, A \rangle$ and $Y = \langle D, A, C, B, C, B, A \rangle$

i \ j	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	2 ↑
5	0							
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3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
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2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	2 ↑
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5	0	1 ↑	1 ↑	1 ↑	2 ↖			
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0	0	0	0	0	0	0	0	0
1	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←	1 ↖	1 ←
2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	2 ↑
5	0	1 ↑	1 ↑	1 ↑	2 ↖	2 ↑	3 ↖	
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0	0	0	0	0	0	0	0	0
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2	0	0 ↑	1 ↖	1 ←	1 ↑	1 ↑	1 ↑	2 ↖
3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	2 ↑
5	0	1 ↑	1 ↑	1 ↑	2 ↖	2 ↑	3 ↖	3 ←
6	0							

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0	0	0	0	0	0	0	0	0
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3	0	0 ↑	1 ↑	1 ↑	2 ↖	2 ←	2 ↖	2 ↑
4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	2 ↑
5	0	1 ↑	1 ↑	1 ↑	2 ↖	2 ↑	3 ↖	3 ←
6	0	1 ↑						

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4	0	1 ↖	1 ↑	1 ↑	2 ↑	2 ↑	2 ↑	2 ↑
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6	0	1 ↑	2 ↖	2 ←	2 ↑	2 ↑	3 ↑	4 ↖

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5	0	1 ↑	1 ↑	1 ↑	2 ↖	2 ↑	3 ↖	3 ←
6	0	1 ↑	2 ↖	2 ←	2 ↑	2 ↑	3 ↑	4 ↖

Longest common subsequence has length 4 and it is ABBA

Pseudocode and analysis

```
LCS-LENGTH(X, Y, m, n)
  let b[1..m, 1..n] and c[0..m, o..n] be new tables
  for i = 1 to m
      c[i, 0] = 0
  for j = 0 to n
      c[0, j] = 0
  for i = 1 to m
      for j = 1 to n
          if x_i == y_j
              c[i, j] = c[i − 1, j − 1] + 1
              b[i, j] = "↖"
          else if c[i − 1, j] ≥ c[i, j − 1]
                  c[i, j] = c[i − 1, j]
                  b[i, j] = "↑"
              else c[i, j] = c[i, j − 1]
                  b[i, j] = "←"
  return c and b
```

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- Time dominated by instructions inside the two nested loops which execute $m \cdot n$ times

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  return c and b
```

- Time dominated by instructions inside the two nested loops which execute $m \cdot n$ times
- Total time is $\Theta(m \cdot n)$.

Pseudocode and analysis for printing solution

```
PRINT-LCS(b, X, i, j)
  if i == 0 or j = 0
      return
  if b[i, j] == "↖"
      PRINT-LCS(b, X, i − 1, j − 1)
      print x_i
  elseif b[i, j] == "↑"
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  else PRINT-LCS(b, X, i, j − 1)
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- Each recursive call decreases $i + j$ by at least one.

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  else PRINT-LCS(b, X, i, j − 1)
```

- Each recursive call decreases $i + j$ by at least one.
- Hence, if we let $n = i + j$, the time needed is at most $T(n) \leq T(n-1) + \Theta(1)$ which is $O(n)$
- We can thus print the found string in time $\Theta(|X| + |Y|)$ (the lower bound following from that $T(n) \geq T(n-2) + \Theta(1)$)

Summary

- Identify choices and optimal substructure
- Write optimal solution recursively as a function of smaller subproblems
- Use top-down with memoization or bottom-up to solve the recursion efficiently (without repeatedly solving the same subproblems)

OPTIMAL BINARY SEARCH TREES

Lecture 11, 25.03.2025

Searching on Facebook

More popular than

Optimal binary search trees

- Given sequence $K = \langle k_1, k_2, \ldots, k_n \rangle$ of n distinct keys, sorted ($k_1 < k_2 < \cdots < k_n$).
- Want to build a binary search tree from the keys

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- Actual cost = # of items examined

  For key k_i, cost = $\text{depth}_T(k_i) + 1$, where $\text{depth}_T(k_i)$ denotes the depth of k_i in BST T

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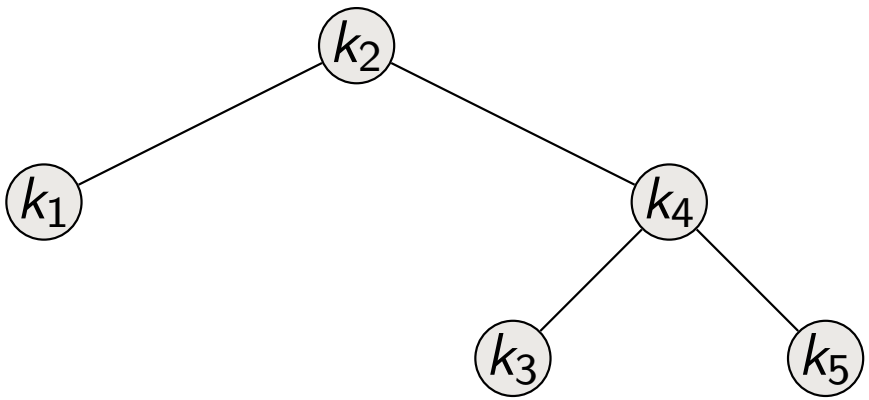
$$\begin{aligned}\mathbb{E}[\text{search cost in } T] &= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) p_i \\ &= 1 + \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i\end{aligned}$$

Example

i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3



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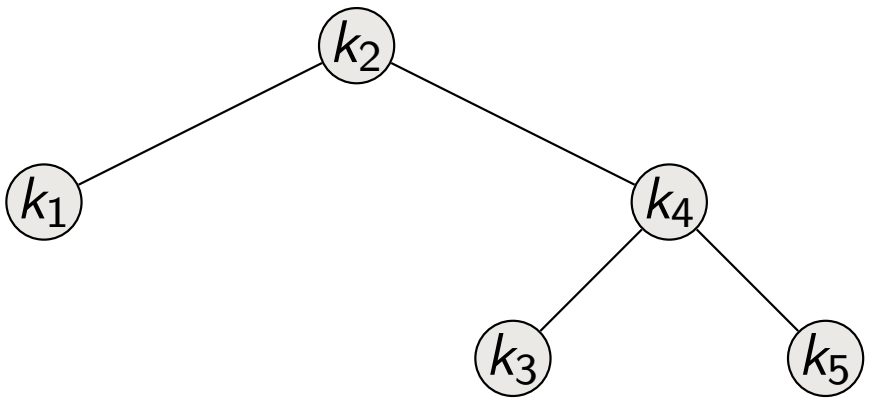
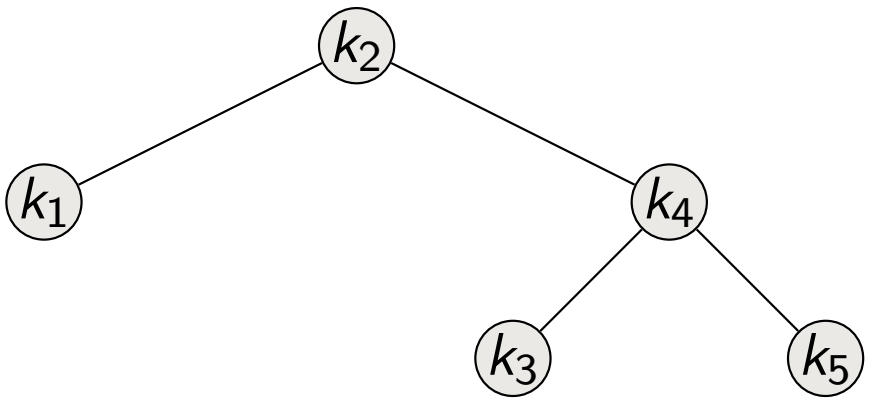


i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	.25
2	0	0
3	2	.1
4	1	.2
5	2	.6
		1.15

Example

i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3



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1	1	.25
2	0	0
3	2	.1
4	1	.2
5	2	.6
		1.15

Therefore, $\mathbb{E}[\text{search cost}] = 2.15$

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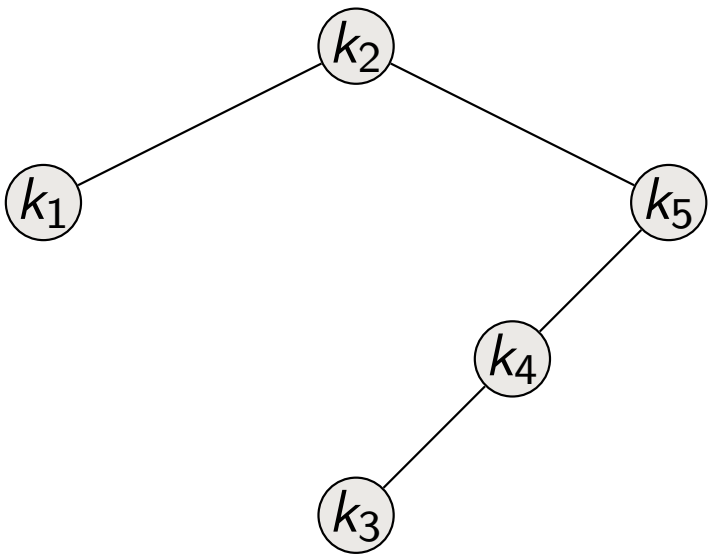

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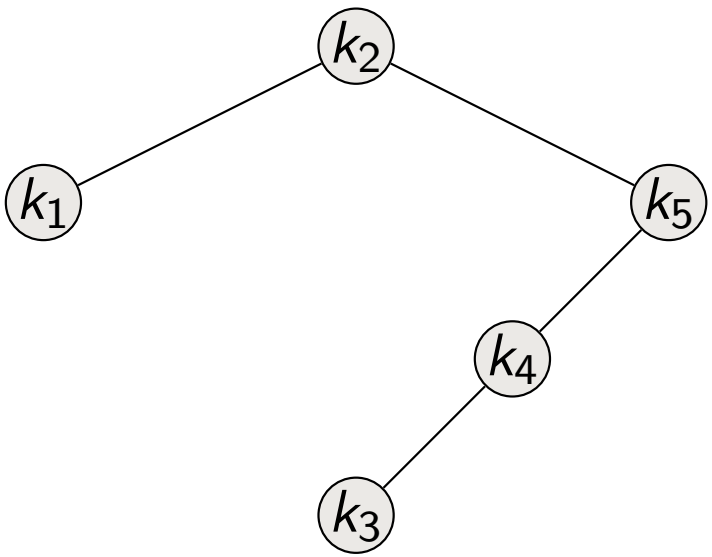

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1	1	.25
2	0	0
3	3	.15
4	2	.4
5	1	.3
		1.10

Example

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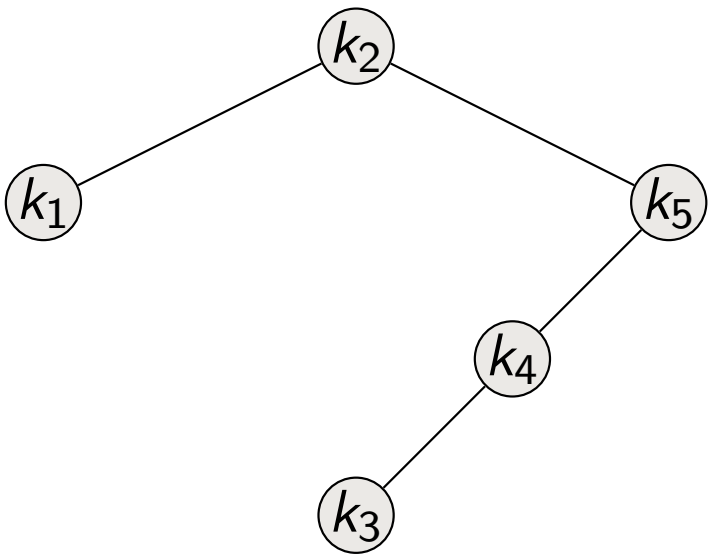

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1	1	.25
2	0	0
3	3	.15
4	2	.4
5	1	.3
		1.10

Therefore, $\mathbb{E}[\text{search cost}] = 2.10$, which turns out to be optimal

Observations

- Optimal BST might not have smallest height
- Optimal BST might not have highest-probability key at root

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Build by exhaustive checking?

- Construct each n-node BST
- For each put in keys
- Then compute expected search cost

Observations

- Optimal BST might not have smallest height
- Optimal BST might not have highest-probability key at root

Build by exhaustive checking?

- Construct each n-node BST
- For each put in keys
- Then compute expected search cost
- But there are exponentially many trees

Observations

- Optimal BST might not have smallest height
- Optimal BST might not have highest-probability key at root

Build by exhaustive checking?

- Construct each n-node BST
- For each put in keys
- Then compute expected search cost
- But there are exponentially many trees

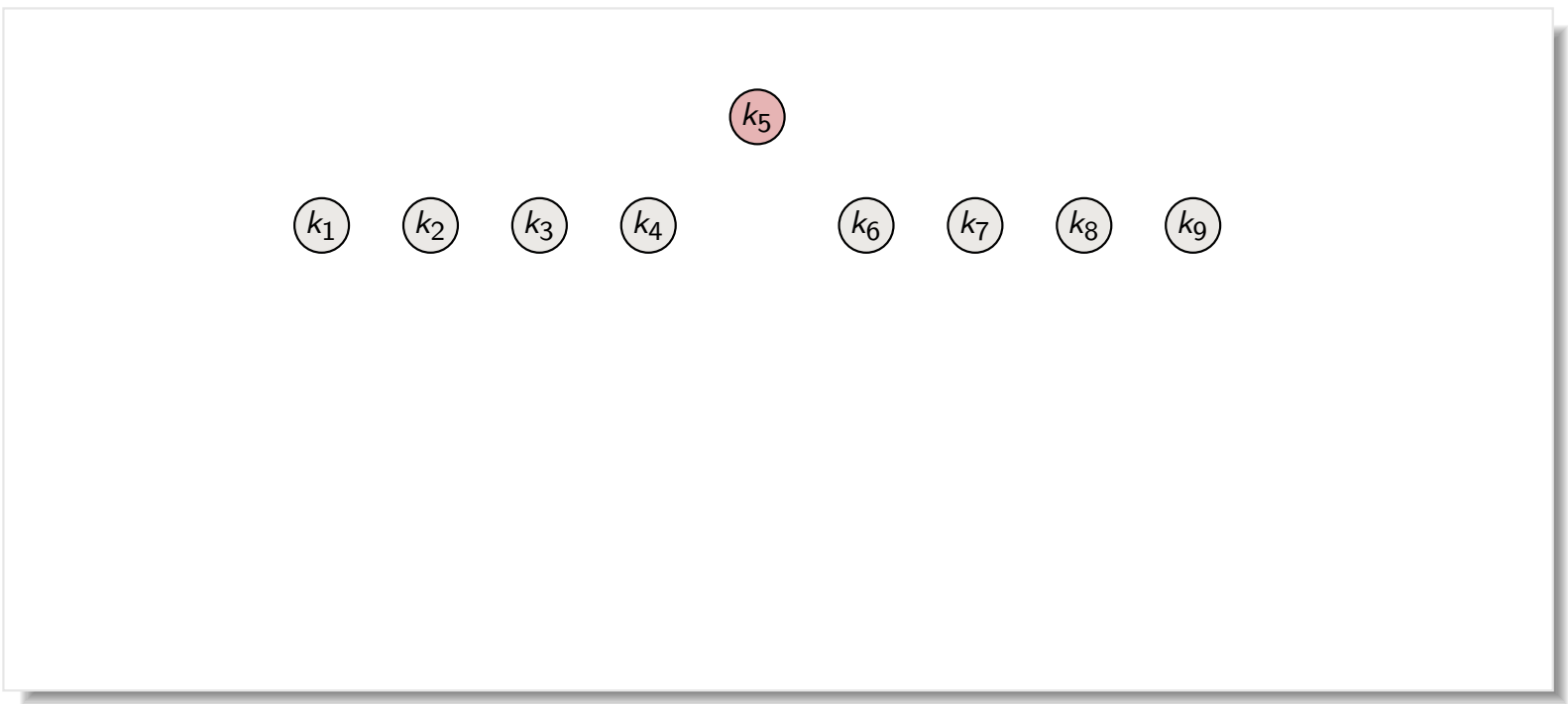
DP comes to the rescue :)

Optimal substructure

A binary search tree can be built by first picking the root and then building the subtrees recursively

Optimal substructure

A binary search tree can be built by first picking the root and then building the subtrees recursively



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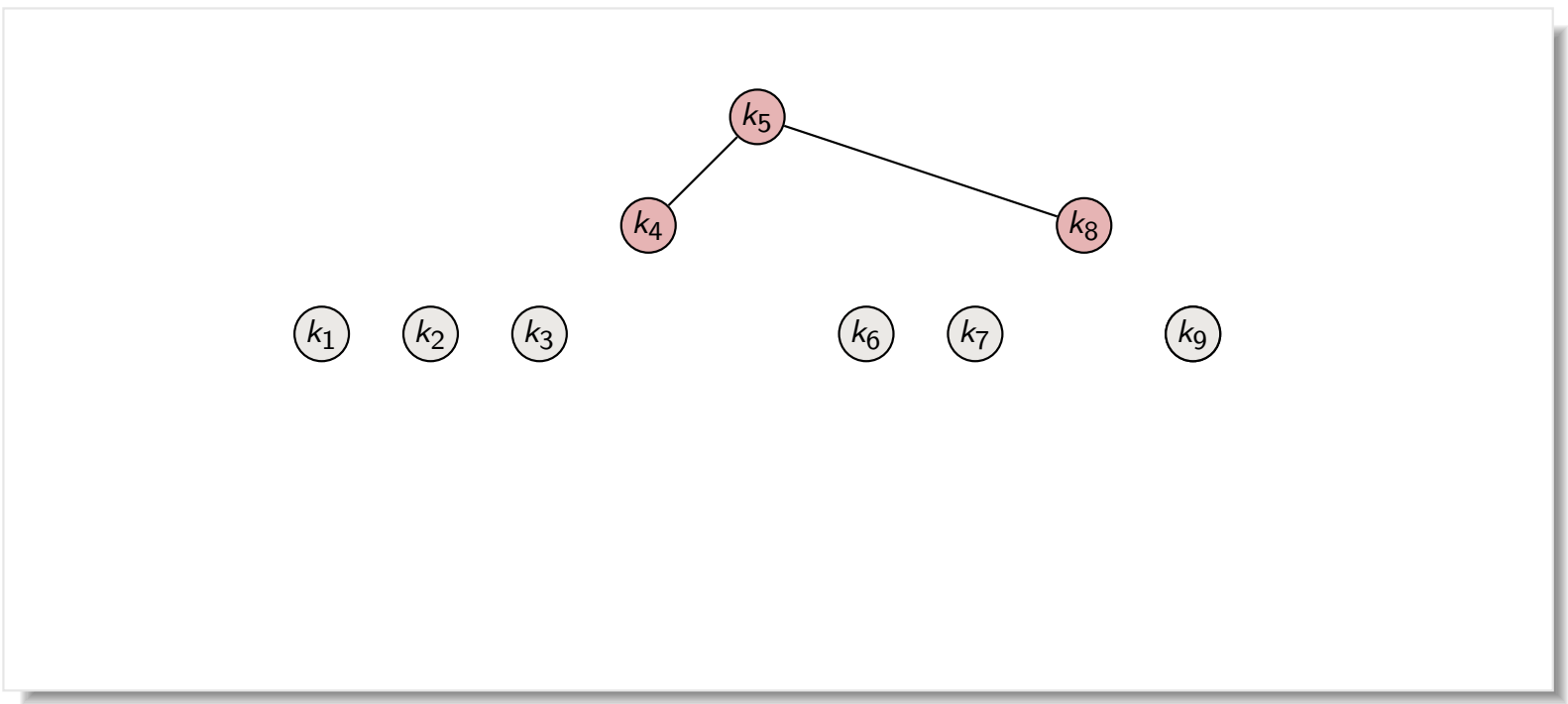

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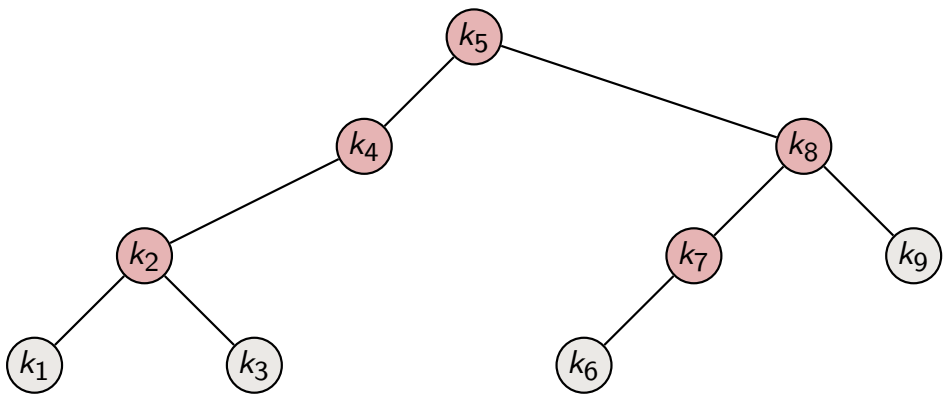

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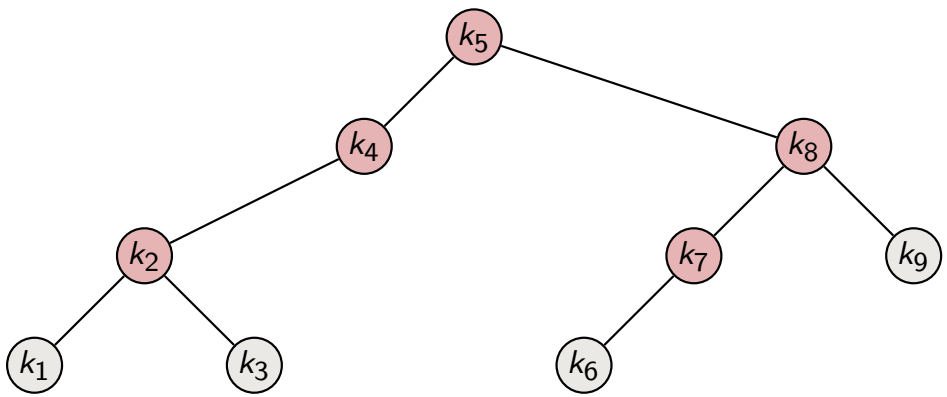

$$\mathbb{E}[\text{search cost}] = p_5 + 2p_4 + 3p_2 + 4p_1 + 4p_3 + 2p_8 + 3p_7 + 3p_9 + 4p_6$$

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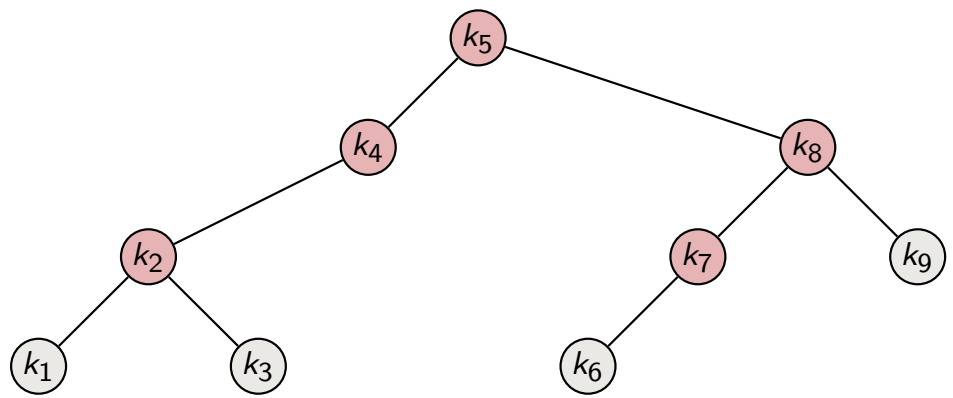


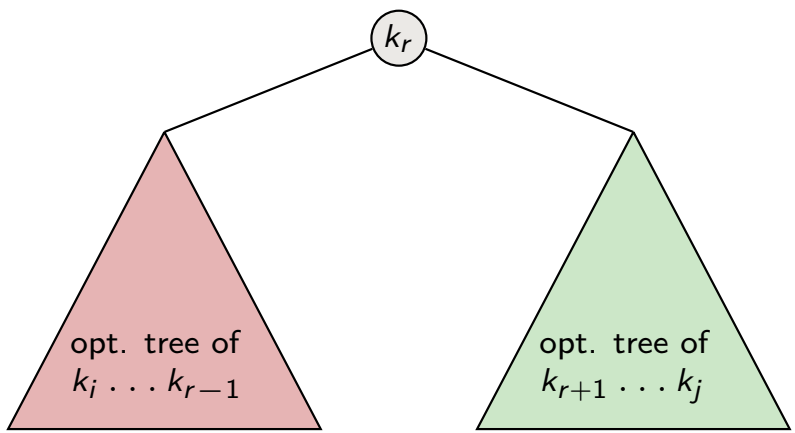
$$\begin{aligned}\mathbb{E}[\text{search cost}] = p_5 \\ &+ p_1 + p_2 + p_3 + p_4 + \mathbb{E}[\text{search cost left subtree}] \\ &+ p_6 + p_7 + p_8 + p_9 + \mathbb{E}[\text{search cost right subtree}]\end{aligned}$$

Optimal substructure

A binary search tree can be built by first picking the root and then building the subtrees recursively

After picking root solution to subtrees must be optimal

Build tree of nodes $k_i < k_{i+1} < \cdots < k_{j-1} < k_j$ by selecting best root r:



$\mathbb{E}[\text{search cost}] = p_r$

$+p_i + \cdots + p_{r-1} + \mathbb{E}[\text{search cost left subtree}]$

$+p_{r+1} + \cdots + p_j + \mathbb{E}[\text{search cost right subtree}]$

Recursive formulation

- Let $e[i, j] =$ expected search cost of optimal BST of $k_i \ldots k_j$

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$$e[i,j] = \begin{cases} 0 & \text{if } i = j+1 \\ \min_{i \le r \le j}\{e[i, r-1] + e[r+1, j] + \sum_{\ell=i}^{j} p_\ell\} & \text{if } i \le j \end{cases}$$

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- Solve using bottom-up or top-down with memoization

Bottom-up example

i	1	2	3	4	5
p_i	.25	.2	.05	.2	.3

$$e[i,j] = \begin{cases} 0 & \text{if } i = j+1 \\ \min_{i \le r \le j}\{e[i,r-1] + e[r+1,j] + \sum_{\ell=i}^{j} p_\ell\} & \text{if } i \le j \end{cases}$$

e	0	1	2	3	4	5
1						
2						
3						
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5						
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e	0	1	2	3	4	5
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

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1	0	.25				
2		0				
3			0			
4				0		
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e	0	1	2	3	4	5
1	0	.25	.65			
2		0	.2	.3		
3			0	.05	.3	
4				0	.2	.7
5					0	.3
6						0

Bottom-up example

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$$e[i,j] = \begin{cases} 0 & \text{if } i = j+1 \\ \min_{i \le r \le j}\{e[i,r-1] + e[r+1,j] + \sum_{\ell=i}^{j} p_\ell\} & \text{if } i \le j \end{cases}$$

e	0	1	2	3	4	5
1	0	.25	.65	.8		
2		0	.2	.3		
3			0	.05	.3	
4				0	.2	.7
5					0	.3
6						0

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e	0	1	2	3	4	5
1	0	.25	.65	.8		
2		0	.2	.3	.75	
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e	0	1	2	3	4	5
1	0	.25	.65	.8		
2		0	.2	.3	.75	
3			0	.05	.3	.85
4				0	.2	.7
5					0	.3
6						0

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e	0	1	2	3	4	5
1	0	.25	.65	.8	1.25	
2		0	.2	.3	.75	
3			0	.05	.3	.85
4				0	.2	.7
5					0	.3
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e	0	1	2	3	4	5
1	0	.25	.65	.8	1.25	
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4				0	.2	.7
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e	0	1	2	3	4	5
1	0	.25	.65	.8	1.25	2.1
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e	0	1	2	3	4	5
1	0	.25	.65	.8	1.25	2.1
2		0	.2	.3	.75	1.35
3			0	.05	.3	.85
4				0	.2	.7
5					0	.3
6						0

Optimal BST has expected search cost 2.1
Can save decisions to reconstruct tree

Pseudocode of bottom-up

```
OPTIMAL-BST(p, q, n)
  let e[1..n+1, 0..n], w[1..n+1, 0..n], and root[1..n, 1..n] be new tables
  for i = 1 to n + 1
      e[i, i − 1] = 0
      w[i, i − 1] = 0
  for l = 1 to n
      for i = 1 to n − l + 1
          j = i + l − 1
          e[i, j] = ∞
          w[i, j] = w[i, j − 1] + p_j
          for r = i to j
              t = e[i, r − 1] + e[r + 1, j] + w[i, j]
              if t < e[i, j]
                  e[i, j] = t
                  root[i, j] = r
  return e and root
```

$e[i,j]$ records the expected search cost of optimal BST of $k_i, \ldots, k_j$

$r[i,j]$ records the best root in optimal BST of $k_i, \ldots, k_j$

$w[i,j]$ records $\sum_{\ell=i}^{j} p_\ell$

Runtime Analysis

```
OPTIMAL-BST(p, q, n)
  let e[1..n+1, 0..n], w[1..n+1, 0..n], and root[1..n, 1..n] be new tables
  for i = 1 to n + 1
      e[i, i − 1] = 0
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  for l = 1 to n
      for i = 1 to n − l + 1
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          w[i, j] = w[i, j − 1] + p_j
          for r = i to j
              t = e[i, r − 1] + e[r + 1, j] + w[i, j]
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  return e and root
```

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```

- Runtime dominated by three nestled loops: total time is $\Theta(n^3)$

Runtime Analysis

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  let e[1..n+1, 0..n], w[1..n+1, 0..n], and root[1..n, 1..n] be new tables
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      for i = 1 to n − l + 1
          j = i + l − 1
          e[i, j] = ∞
          w[i, j] = w[i, j − 1] + p_j
          for r = i to j
              t = e[i, r − 1] + e[r + 1, j] + w[i, j]
              if t < e[i, j]
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```

- Runtime dominated by three nestled loops: total time is $\Theta(n^3)$
- Alternatively, $\Theta(n^2)$ cells to fill in

Runtime Analysis

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  for l = 1 to n
      for i = 1 to n − l + 1
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  Most cells take $\Theta(n)$ time to fill in

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                  root[i, j] = r
  return e and root
```

- Runtime dominated by three nestled loops: total time is $\Theta(n^3)$
- Alternatively, $\Theta(n^2)$ cells to fill in
  Most cells take $\Theta(n)$ time to fill in
  Hence, total time is $\Theta(n^3)$

Summary

- Identify choices and optimal substructure
- Write optimal solution recursively as a function of smaller subproblems
- Use top-down with memoization or bottom-up to solve the recursion efficiently (without repeatedly solving the same subproblems)
- Do a lot of exercises!